Grade 12 Math Review Workbook

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Preliminary Concepts

Radians

The radian unit uses arc length to talk about angles, meaning that angles are defined by the arc length that they create. For any given circle, an angle of 1 radian will create an arc length equal to the radius of the circle. If s is the arc length, the following rule applies

$$\frac{s}{r} = \theta$$

It takes exactly 2π many radii to complete a circle, meaning that $360°$ is the same as 2π radians.

Example: If the central angle of a circle is 80° and the radius is 9cm, what is the arc length?

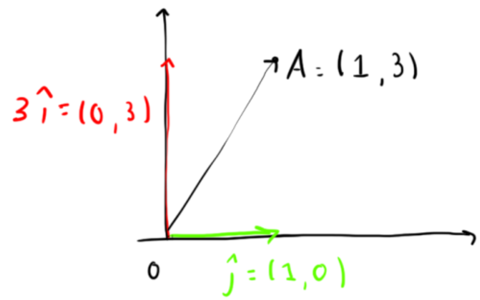
Vectors

A vector is best thought of as something with both a magnitude and a direction, whereas a number just has a magnitude. There are two common ways to think about vectors

1. With an explicit direction and magnitude
   a. Ex) A wind travelling at 100km/h with a bearing of 260°
2. Using their components
   a. Ex) A = (3, 5)

Vector Components

In the cartesian plane there are two basis vectors, $\hat{\imath} = (1,0)$ and $\hat{\jmath} = (0,1)$. $\hat{\imath}$ is parallel to the x-axis and $\hat{\jmath}$ is parallel to the y-axis. Every vector is made with these vectors. The components of any vector tell us how much to scale each basis vector.

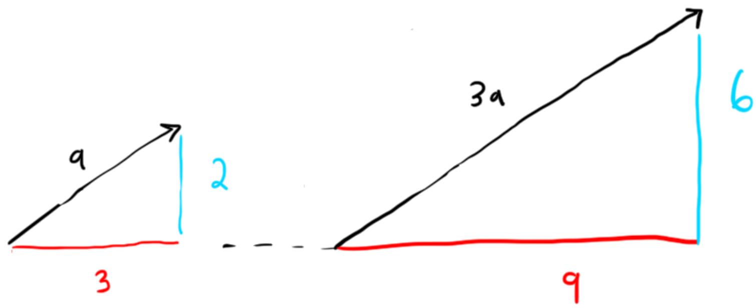


Example: Vector A has a length of 7, and it is 67° from the horizontal axis, going counter-clockwise. What are its components?

Basic Vector Operations

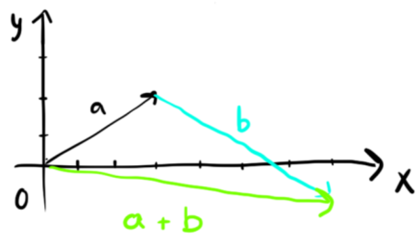
Scalar Multiplication

Scalar multiplication involves scaling each component of the vector. If $\vec{a} = (3,2)$ then $3\vec{a} = (9,6)$. Graphically, scalar multiplication means scaling the length of the vector by the scaling factor.



Addition

Vector addition involves adding the vectors component-wise. For example, if $\vec{a} = (3,2)$ and $\vec{b} = (4,-3)$, then $\vec{a} + \vec{b} = (3+4, 2-3) = (7,-1)$. Graphically, vector addition means adding the vectors tip to tail.

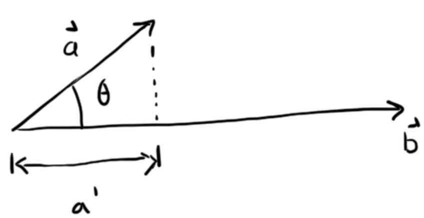


Magnitude

If $\vec{a} = (x, y)$ then $|\vec{a}| = \sqrt{x^2 + y^2}$. As you can tell, this formula is a direct result of the Pythagorean Theorem.

Dot Product

The dot product is an operation between two vectors. When understanding the dot product, is it helpful to think about it geometrically. For some vector a and b, it is computed as the product of their magnitudes, but one vector is projected onto the other. The result of the dot product is a scalar, not a vector.



If a' is the projection of $\vec{a}$ onto $\vec{b}$, then $\vec{a} \cdot \vec{b} = a' \cdot |\vec{b}|$. Generally, the dot product is known as $|\vec{a}||\vec{b}|cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$. Mathematically, the dot product is computed as $a_x b_x + a_y b_y$.

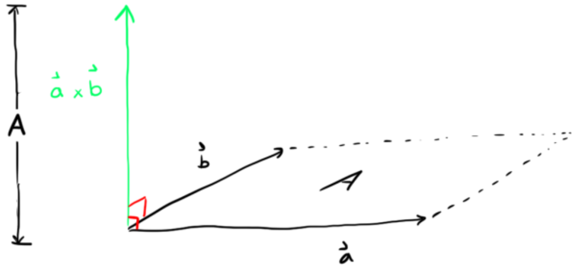
Example: Compute the angle between $\vec{a} = (1, 4)$ and $\vec{b} = (7, 6)$

Example: Find the length of $\vec{a} = (1, 4)$ projected onto $\vec{b} = (7, 6)$

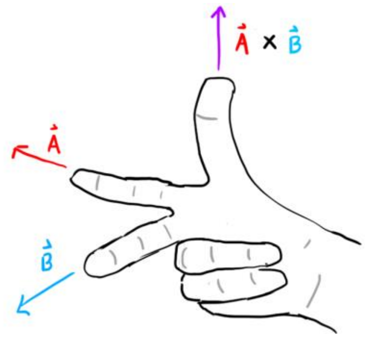
A common use of the dot product is to determine if two vectors are perpendicular. If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are perpendicular. This is because if the result of the dot product is zero, there will have been no length projected onto the other vector. If there is no length projected onto the other vector, it must be the case that the two vectors are perpendicular.

Cross Product

The cross product is just like the dot product, it is an operation between two vectors. The key difference is that it only works between 3+ dimensional vectors (i.e. vectors with three components or more). The operation finds the vector perpendicular to both input vectors with a magnitude equal to the area the input vectors make. The magnitude of the result $|\vec{a}\ x\ \vec{b}| = |\vec{a}||\vec{b}|sin\theta$.



Note that for any two vector inputs, there is always two possible vector directions that are perpendicular to the input vectors. The correct direction conforms to the right-hand rule:



The process for computing the cross product given the components of both input vectors is as follows.

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ then the cross product can be expanded using distributivity.

$$\vec{a}\ x\ \vec{b} = a_1b_1(\hat{\imath}\ x\ \hat{\imath}) + a_1b_2(\hat{\imath}\ x\ \hat{\jmath}) + a_1b_3(\hat{\imath}\ x\ \hat{k}) +$$

$$a_2b_1(\hat{\jmath}\ x\ \hat{\imath}) + a_2b_2(\hat{\jmath}\ x\ \hat{\jmath}) + a_2b_3(\hat{\jmath}\ x\ \hat{k}) +$$

$$a_3b_1(\hat{k}\ x\ \hat{\imath}) + a_3b_2(\hat{k}\ x\ \hat{\jmath}) + a_3b_3(\hat{k}\ x\ \hat{k})$$

Example: Using distributivity, compute the cross product between $\vec{\boldsymbol{a}} = (\mathbf{3}, \mathbf{4}, \mathbf{5})$ and $\vec{\boldsymbol{b}} = (\mathbf{1}, -\mathbf{3}, \mathbf{5})$

Parametric Equation for a Line

The blueprint for a line is the following

$$\vec{r} = \vec{r}_0 + t\vec{d}$$

Where $\vec{r}_0$ is the starting point, $\vec{d}$ is the direction vector, and t is a parameter.

Example: Construct a line with a slope of 3 that passes through $(2, 3, 7)$ and is parallel to the xy plane.

Equation of a Plane

If $\vec{n}$ is the normal of a plane centered at the origin, then the equation of the plane is,

$$\vec{n} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 => \; n_x x + n_y y + n_z z = 0$$

A plane not centered at the origin has the following formula.

$$\vec{n} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0 => n_X x + n_y y + n_z z = C$$

A plane is defined by its normal - every point on the plane is perpendicular to the normal. The formula says that the dot product between the normal and any point equals zero. This means that the only points that will satisfy the equation will be on the plane since they will be perpendicular to the normal.

Example: What is the formula for a plane centered at (3,0,0) parallel with the zy plane?

Example: Two vectors $\vec{a} = (1, 0, 0)$ and $\vec{b} = (0, 1, 1)$ both live on a plane, what is the formula of the plane?

Limits and Continuity

Evaluating a limit of a function means asking what happens when x approaches some value. A limit of a function is designated by the following,

$$f(x) \to L \; as \; x \to a, \lim_{x \to a} f(x) = L$$

A limit L of a function at a point x exists if no matter how x is approached, the function will always approach L. This means that the left- and right-hand limits equal each other, and that they both exist.

$\boldsymbol{\lim_{x \to a} f(x) = L}$ if and only if $\lim_{x \to a^-} f(x)$ exists, $\lim_{x \to a^+} f(x) = L$ exists, and $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

Example: Compute the following limits

a) $\lim_{x \to 1} g(x), g(x) = \frac{1}{1-x}$.

b) $\lim_{x \to 5} x^2 + 3x$

c) $\lim_{x \to \infty} \frac{1}{x}$

Continuity

A function f(x) is continuous at a point x = a if it meets the following set of criteria:

1. $\lim_{x \to a} f(x)$ exists
2. $f(a)$ exists (The limit at that point exists)
3. $\lim_{x \to a} f(x) \; = f(a)$

A function is defined to be continuous if for all x in the domain of the function, the function is continuous.

Example: Let $\boldsymbol{f(x) = \begin{cases} x^2 & , x \leq 2 \\ -2x + 8 & , x > 2 \end{cases}}$ Is f(x) continuous at $\boldsymbol{x = 2}$?

Transformations of Functions

Recall the mapping rule

$$y = af\big(k(x-d)\big) + c$$

When you look at a function, you can think of it as conforming to this syntax. From the syntax, you can pull out the variables to construct a cohesive description of the transformation.

For example, given $y = \frac{1}{2}e^{2x-3}$+4,

1. $a = \frac{1}{2}$
2. $c = 4$
3. $k = 2$
4. $d = \frac{3}{2}$
5. $f(x) = e^x$

Each variable has a specific effect on the function. The effects are as follows,

1. a is the vertical stretch factor
2. c describes the vertical translation
3. $\frac{1}{k}$ is the horizontal stretch factor
4. d describes the horizontal translation
5. $f(x)$ is the original, *untransformed* function

When graphing the function, we can convert each point as follows,

x	y
…	…
…	…
…	…

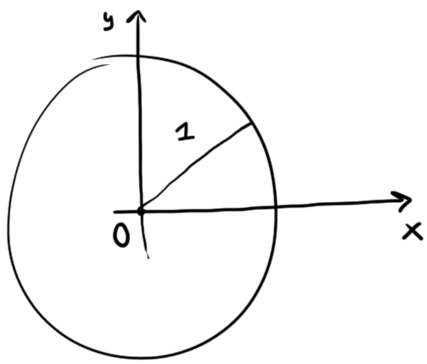
$\frac{x}{k} + d$	$ay + c$
…	…
…	…
…	…

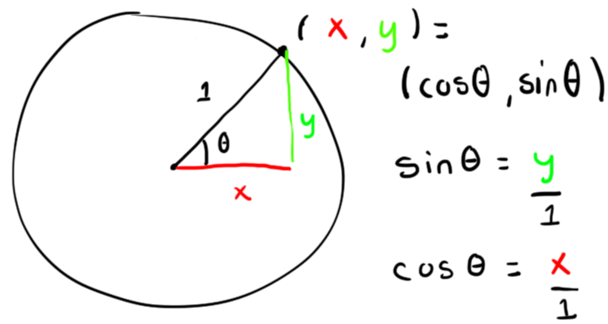
Notice that the input to $f(x)$ is $k(x-d)$. The x variable is transformed before it enters as the input. Let us call the final input x'. When graphing f(x), we plot each y coordinate at x, not x'. In the table of values on the right, x' is reversed to get the original x.

Example: Model the height as a function of time of a point on a Ferris wheel that has a diameter of 75 m, has an rpm of 1.5, and whose center is located 100 m of the ground.

Trigonometric Functions

Recall the definition of the unit circle. It is a circle with radius 1 centered at the origin. Analysis using a triangle and the trig ratios reveals that any (x, y) coordinate on the circle is equal to $(cos\theta, sin\theta)$.





It happens to be the case that $sinx$ and $cosx$ are defined through the unit circle. For a point on the unit circle with angle θ between the positive x-axis and the radius to the point, $sin\theta$ is the y coordinate and $cos\theta$ is the x coordinate. $tan\theta$ is the ratio of the y and x coordinates, making $tan\theta = \frac{sin\theta}{cos\theta}$.

Trigonometic Identities

Pythagorean Identity

$$1 = \sin^2(\theta) + \cos^2(\theta)$$

In the unit circle, $y^2 + x^2 = 1$, $sin\theta = y$, and $cos\theta = x$. If you substitute x and y for $cos\theta$ and $sin\theta$, you get the identity.

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = cosx$$

$$\cos\left(\frac{\pi}{2} - x\right) = sinx$$

Double Angle Identities

$$\sin(2x) = 2sinxcosx$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

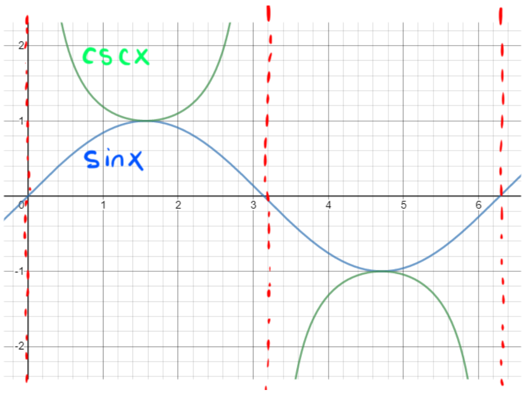
Example: Prove the following identities

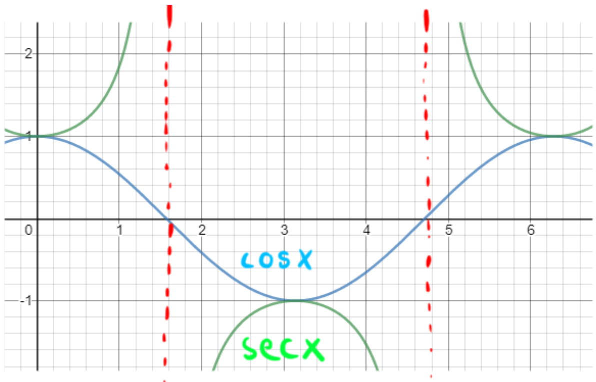
a) $\boldsymbol{tanxsinx + cosx = secx}$

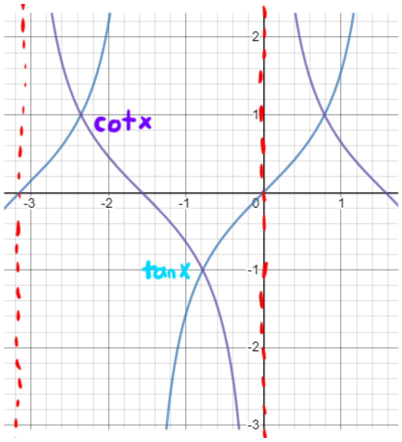
b) $\mathbf{1 - tan^2x = \frac{cos(2x)}{cos^2x}}$

Reciprocal Trig Functions

$$cscx = \frac{1}{sinx}, secx = \frac{1}{cosx}, cotx = \frac{1}{tanx}$$







These functions are best understood as a transformation of the three original functions. Asymptotes from the original function become zeros in the new function and zeros from the original function become asymptotes in the new function. Larger values from the original function will get pinched to the x-axis and smaller values will tend to diverge from the x-axis.

Inverse Functions

Recall that all functions transform some input into an output. An inverse function $f^{-1}(x)$ is directly related to some other function, $f(x)$. $f^{-1}(x)$ does the **opposite** operation to $f(x)$. If you have some number x and you apply $f(x)$ and $f^{-1}(x)$ in series, you will have x at the end. This is seen mathematically as $f^{-1}(f(x)) = x$. In practice, this is commonly done with $y = x^2$ and the $\sqrt{x}$ function when you need to solve for x.

For all inverse functions,

- Every point (x, y) of the original function maps to (y, x) on the inverse function.
- The domain of the original function is the range of the inverse function and the range of the original function is the domain of the inverse function.
- With respect to the original function, the inverse function is reflected over the $y = x$ line.

Example: Find the inverse function to $y = 4(x-3)^4 + 21$

Logarithmic Function

A logarithmic function is the inverse function to an exponential function. The logarithm function answers the question "What exponent do I have to raise the base to in order to get the input, x?". Since each exponential function has a base, each logarithm function will only work for a specific base. This is seen as $\log_a(x)$.

Example: Write $6^2 = 36$ in logarithmic form

Example: Determine the exact value of $\log_{\frac{1}{4}} 16$ without using a calculator

Properties of Logarithmic Functions

1. $\log_a m + \log_a n = \log_a mn$
2. $\log_a m - \log_a n = \log_a \frac{m}{n}$
3. $\log_a m^n = nlog_a m$
4. $\log_a 1 = 0$ (This is the case since any base raised to zero is 1.)
5. $\log_a a = 1$ (This is the case since any base raised to the power of one evaluates to itself.)
6. $a^{\log_a x} = x$ (This is an example of $f^{-1}(f(x)) = x$.)
7. $\log_a x = \frac{\log_b x}{\log_b a}$

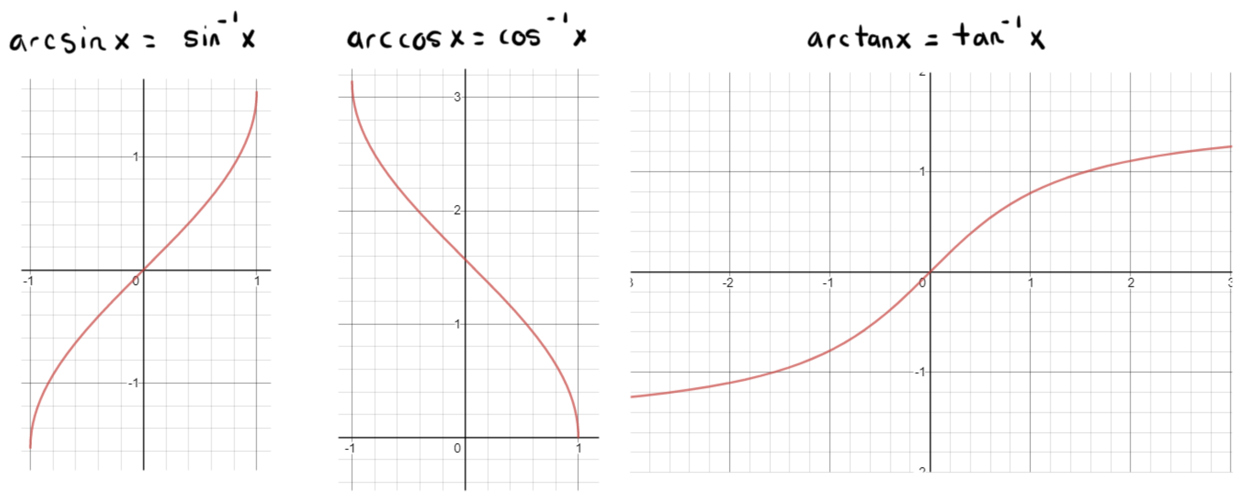
Example: Write in simpler terms

$$\boldsymbol{f(x) = ln(\frac{(x-3)^5}{7})}$$

Example: Write in simpler terms

$$\boldsymbol{f(x) = ln(\frac{x-2}{y^3\sqrt[3]{z}})}$$

Inverse Trigonometric Functions



The three primary inverse trigonometric functions, $arcsinx$, $arccosx$, and $arctanx$ have domains that are specific and important to understand. Recall that the domain of an inverse function is equal to the range of the original function. This means that for both $arcsinx$ and $arccosx$, the domain is $-1 \leq x \leq 1$. For $arctanx$, the domain is infinite since the range of $tanx$ is infinite.

The range for $arcsinx$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, the range for $arccosx$ is $0 \leq y \leq \pi$, and the range for $arctanx$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. The domains of the original functions (i.e. $sinx, cosx, tanx$) have been restricted so that the inverse functions are valid functions. If the domains were not restricted, the inverse functions would not pass the vertical line test.

Rates of Change

A rate of change is fundamentally a rate, which means it is a fixed ratio between two things. In particular, it means observing change over some parameter. For example, the speed of a car going around a turn is 40km/h. This is a measure of change across time, which is a rate of change.

Average Rate of Change

For some function $f(x)$,the average rate of change is calculated as

$$A.R.O.C = \frac{f(a+h) - f(a)}{h}$$

Example: Compute the average rate of change for $f(x) = e^x$ over $0 \le x \le 1$

Instantaneous Rate of Change

An instantaneous rate of change is the rate of change measured at a single point. How can you have a rate of change at a single point? You need at least 2 points to measure change. Well, this is very true. The speedometer on your car merely estimates the I.R.O.C. It does this by computing the average rate of change over a very small interval. For functions, this can be done using tangent lines. A tangent line intersects the function at only one point (around a small interval near the intersection point) and has a slope equal to the value of the rate of change at that point. If we draw a tangent line and compute its slope, we will get the value of the instantaneous rate of change.

Example: Estimate the I.R.O.C of $f(x) = x^2$ at x=2.

Derivatives

In the last section we stumbled upon the idea of a derivative. The not-estimated, actual I.R.O.C ***is*** the derivative. A derivative is a function that gives the instantaneous rate of change for every point in the original function domain. It turns out that if we keep making the interval of change smaller and smaller, we get closer and closer to the actual derivative. In fact, if we use limits, we can compute the exact derivative. This is the limit definition of the derivative,

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note that dx is a small, non-zero value that happens to approach zero. The small change to x, dx, is what causes the small change to y, dy. For example, for the function $f(x) = x^2$,

$$\frac{dy}{dx} = 2x, dy = 2xdx$$

Derivative Laws

Using the limit definition to compute derivatives for different functions quickly becomes challenging. There are many derivative laws to speed up the process.

Basic Laws

1. The derivative of a constant is zero, because it never changes.
2. The derivative of $f(x) = g(x) + h(x)$ is $f'(x) = g'(x) + h'(x)$

Power Law

$f(x) = x^n$

$\frac{dy}{dx} = nx^{n-1}$

Product Rule

$f(x) = g(x)h(x)$

$f'(x) = g'(x)h(x) + g(x)h'(x)$

Quotient Rule

$y = \frac{f(x)}{g(x)}$

$\frac{dy}{dx} = \frac{[f'(x)g(x) - f(x)g'(x)]}{[g(x)]^2}$

Chain Rule

$y = g(f(x))$

$\frac{dy}{dx} = g'\big(f(x)\big)f'(x)$

Leibniz's Notation

The chain rule is elegant when working with it using Leibniz's notation.

Consider the following example:

$$y = u^2, u = e^x$$

$$\frac{dy}{du} = 2u, \frac{du}{dx} = e^x$$

$$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx} = 2ue^x = 2e^{2x}$$

Table of Derivatives

$log_a x$	$\frac{1}{xln(a)}$
e^x	e^x
a^x	$a^x lna$
$arcsinx$	$\frac{1}{\sqrt{1-x^2}}$
$arccosx$	$\frac{-1}{\sqrt{1-x^2}}$
$arctanx$	$\frac{1}{1+x^2}$
$sinx$	$cosx$
$cosx$	$-sinx$
$tanx$	$\sec^2(x)$

Example: Differentiate the following

a) $\boldsymbol{f(x) = sin^2(x)}$

b) $\boldsymbol{f(x) = e^{2x}x^4}$

c) $\boldsymbol{\frac{2x^3}{2x-5x^2}}$

Antiderivatives

An antiderivative is the original function of the derivative. When you want to find the antiderivative for a given function, you assume that it is the derivative to some other function, then apply the derivative laws in reverse. Note that there will be a "+ C" added to the end of the antiderivative. This is because when you take the derivative of constants, they evaluate to zero, meaning that there might be an unknown constant in the antiderivative.

Example: Find the antiderivative of $\boldsymbol{f(x) = -2sinxcosx}$

Example: A cannon ball is released from the ground at a speed of 50m/s with an angle θ. Use antiderivatives to find the equation for the height of the disk as a function of time.

Implicit Differentiation

Implicit differentiation is a technique for differentiating when it is difficult to isolate one variable. It involves taking the derivative of both sides, but not in the traditional sense. It means to compute the total change for both sides using the derivative laws, then using algebra to pull out the derivative. This works because for the functions to remain equal they must change at the same rate.

For example, consider $x^2 + y^2 = 9$. If you use derivative laws to compute the small change for both sides, you get $2xdx + 2ydy = 0$. This means that the derivative is then $\frac{dy}{dx} = -\frac{x}{y}$.

Example: Use implicit differentiation to find the derivative $\frac{dy}{dx}$ of $xy - y = 3$

Derivative Applications / Techniques

Related Rates

This technique is all about using one rate to find another rate (hence the fact that the rates are related). It is accomplished by using both the chain rule and implicit differentiation.

Example: An oil refining facility has sprung a leak and is now spilling oil onto a nearby pond. The oil is leaking out a constant rate of 5 liters per second, forming a slick that is roughly circular in shape and 0.0025 meters thick. Include units in your answers. Reminder: there are 1000 liters in a cubic meter

a) **Assuming that at t = 0 there was no oil in the slick, how long does it take for the radius of the slick to reach 70 meters?**

a) **At what rate is the radius of the slick increasing when the radius is 70 meters?**

Optimization

Optimization involves finding the points at which a function is at its largest or its least. To find a minimum or maximum of a function, take the derivative of the function and set the derivative to zero.

Example: Find when $f(x) = 2x^2 - 3x - 4$ is at its smallest.

Second Derivative Test

Sometimes a function might have more than one maximum and minimum. To determine if the critical point is a maximum or a minimum, the second derivative test is used. The second derivative of a function talks about how the derivative is changing. For a maximum, the derivative is decreasing (negative second derivative), and for a minimum, the derivative is increasing (positive second derivative).

Example: Find all the critical points of $f(x) = x^3 - 3x$ and classify each as a minimum or a maximum.

Logarithmic Differentiation

This is a technique for differentiating when the function is too complex to differentiate otherwise. It is done by taking the logarithm of both sides and simplifying using log rules.

Example: Differentiate the following

$$\boldsymbol{y = \frac{x^2}{(1-9x)\sqrt{(x^2+1}}}$$

L'Hôpital's rule

When the following conditions are met, we can use L'Hôpital's rule to compute the limit.

$$\lim_{x \to a} \frac{g(x)}{f(x)} = \frac{0}{0} \; or \; \frac{\pm\infty}{\pm\infty}$$

To evaluate a limit like this we do the following:

$$\lim_{x \to a} \frac{g(x)}{f(x)} = \lim_{x \to a} \frac{g'(x)}{f'(x)}$$

Note that you can apply the rule in series if necessary, so long as each instance meets the condition of use.

Example: Compute the limit,

$$\lim_{x \to -\infty} \frac{x^2}{e^{1-x}}$$

Linear Approximations

Linear approximations are used to estimate the values of a function near some point. A linear approximation uses a tangent line to estimate the y-values. The formula is as follows,

$$L(x) = f(a) + f'(a)(x - a)$$

Example: Find the linearization of $f(x) = 2x^3$ at $a = 2$ and use it to estimate $f(x)$ at $x = 2.01$

References

Some material from this workbook has been sourced from the 2016 APSC 171 final, which can be found in ExamBank under QSPACE.

Some material in this workbook has been adapted from previous workbooks for APSC 171.